University of BRISTOL

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MODELLISTICA E CALCOLO SCIENTIFICO
MOX
MODELING AND SCIENTIFIC COMPUTING

POLITECNICO DI MILANO

PDE REGULARIZED BLOOD VELOCITY ESTIMATION

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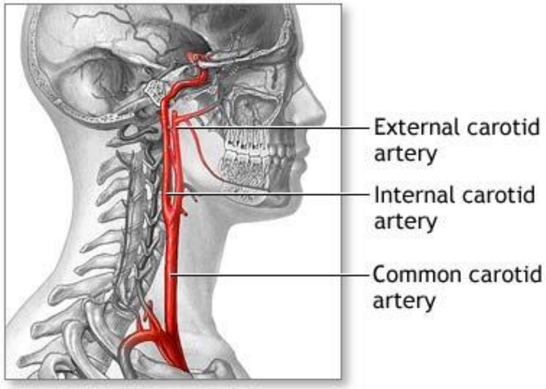
September 12TH 2012

FUTURO IN RICERCA

SNAPLE

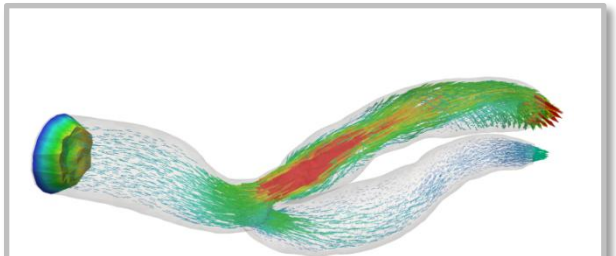
This study is carried out within the project
MACAREN@MOX
MAthematics for CARotid ENdarterectomy@MOX



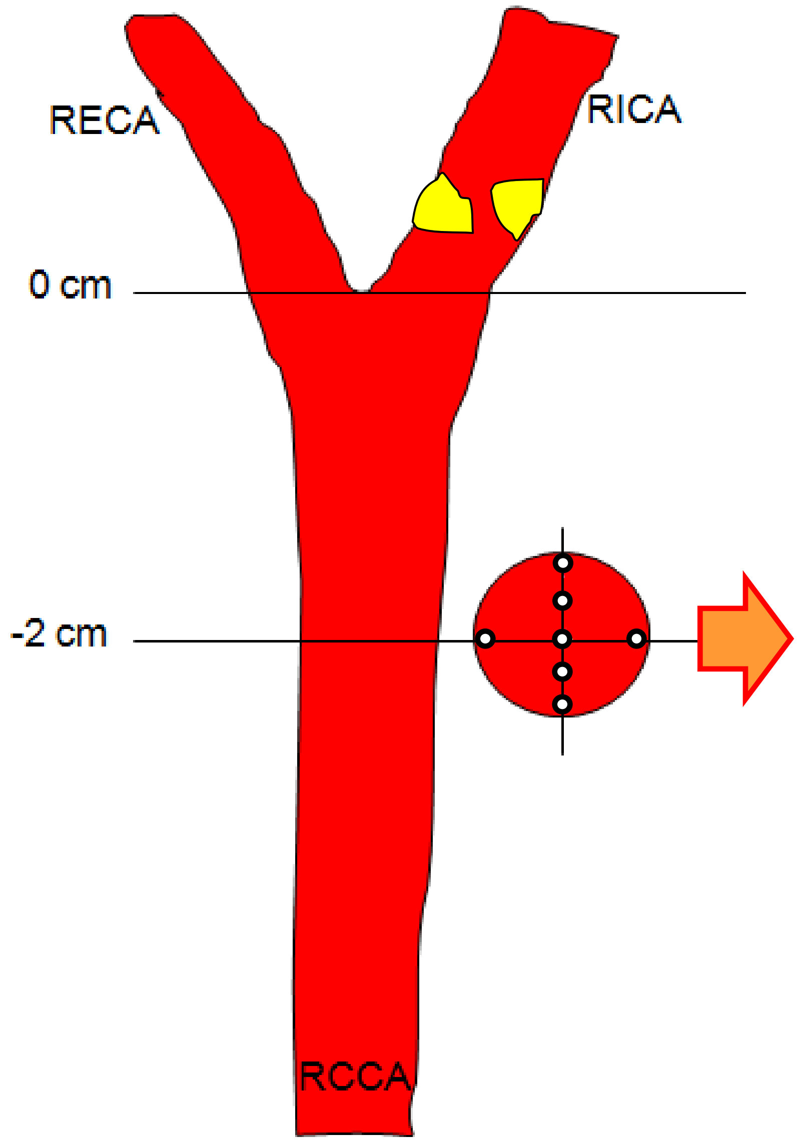
MOTIVATION OF THE MACAREN@MOX PROJECT:

Explain the presence and the histological properties of atherosclerotic plaques in the carotid using:

- geometry of the carotid bifurcation;
- velocity field in the artery
  - real data: eco-doppler measurements;
  - patient specific haemodynamic simulations.


Nobile, Pozzoli, Vergara

AIM: Estimation of the blood-velocity field over a section of the carotid artery using eco-doppler data

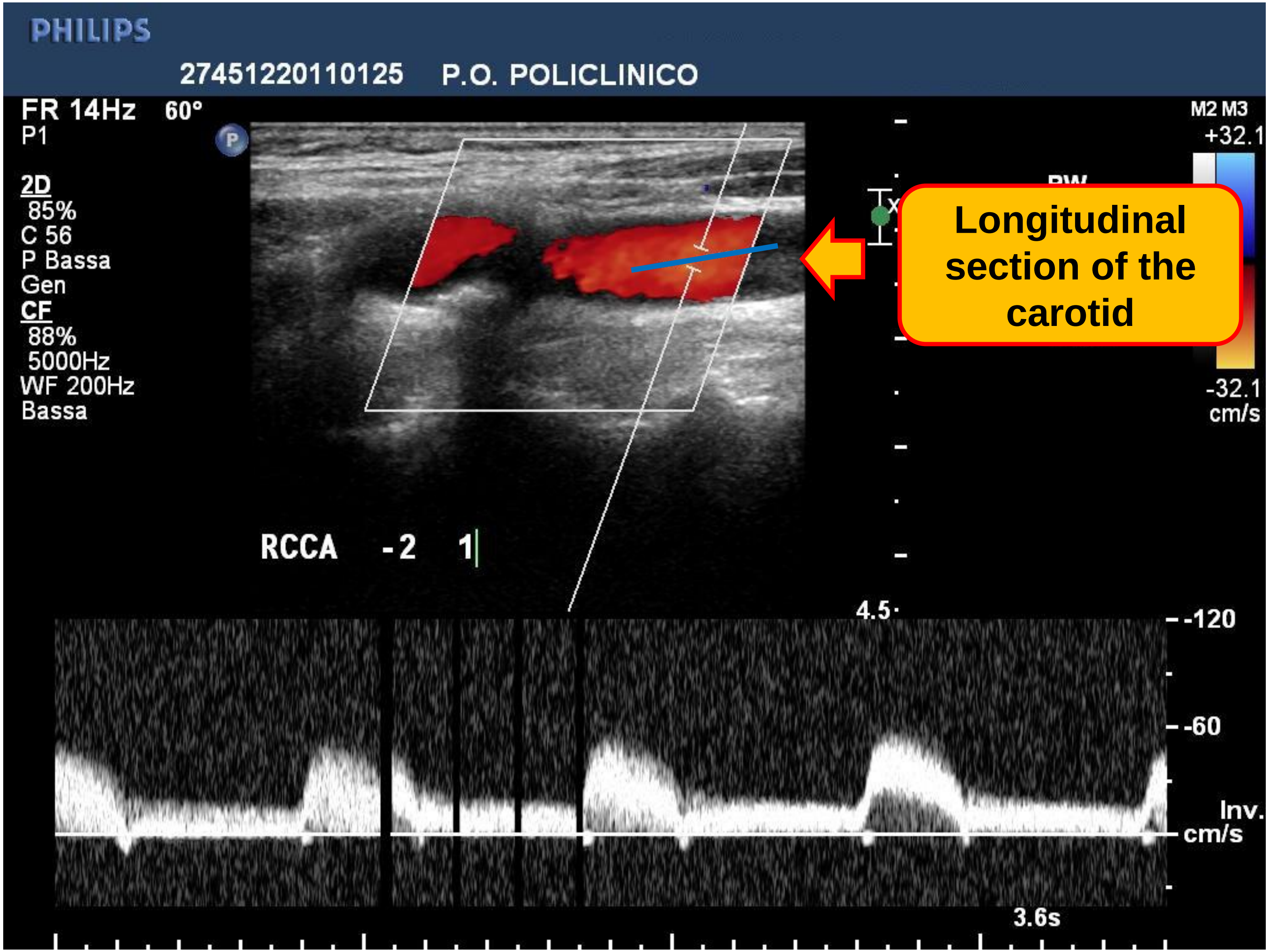


Patients with an high grade stenosis (>70%) that will undergo the removal of the carotid plaque through Thromboendarterectomy (TEA).

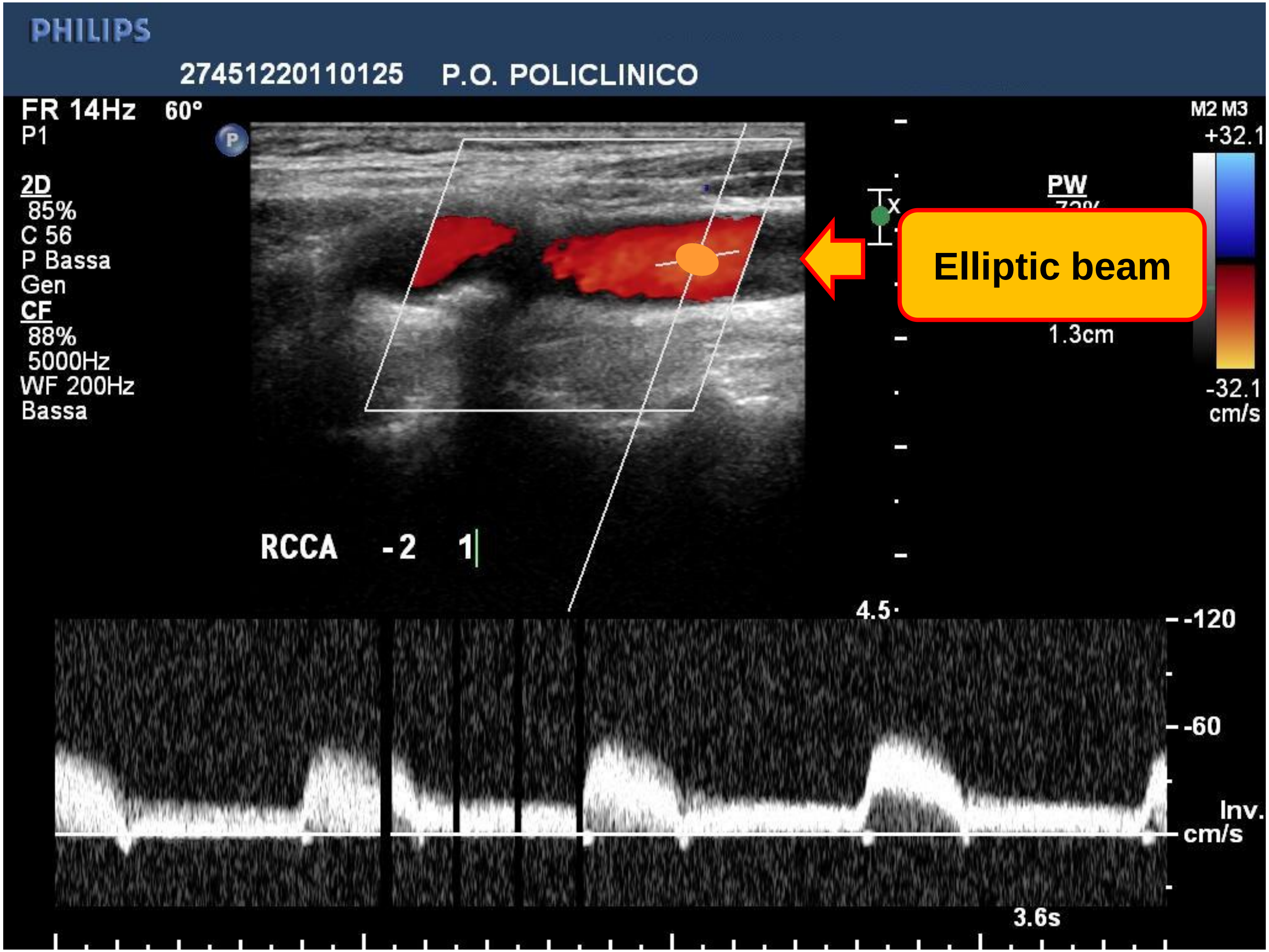
For each patient we consider 7 ecodoppler measurements over the carotid section located 2 cm before the carotid bifurcation.

The eco doppler images are collected by Maurizio Domanin at the Unità Operativa di Chirurgia Vascolare Fondazione I.R.C.C.S. Cà Granda Ospedale Maggiore Policlinico Milano

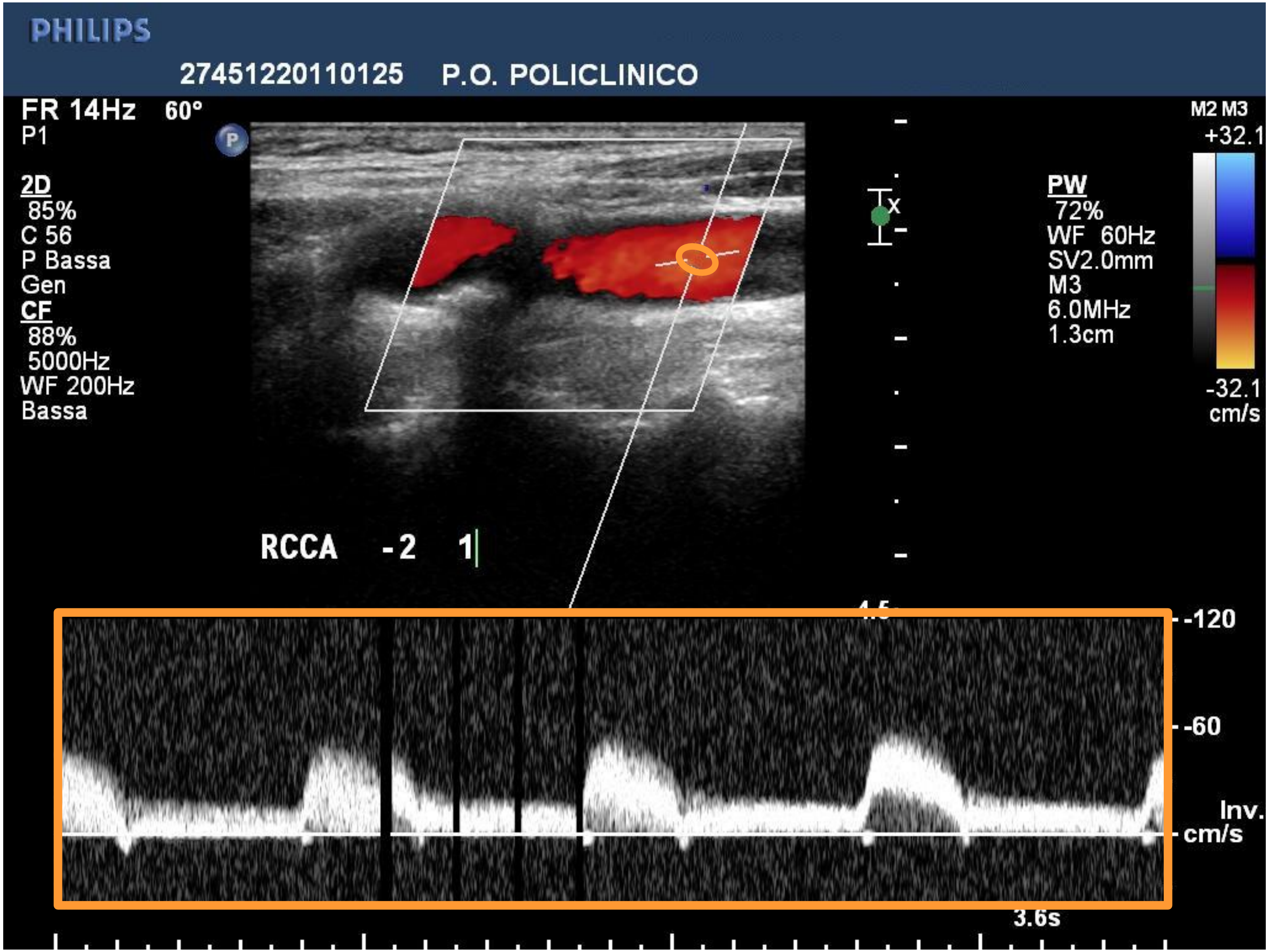
MOTIVATING PROBLEM: DATA



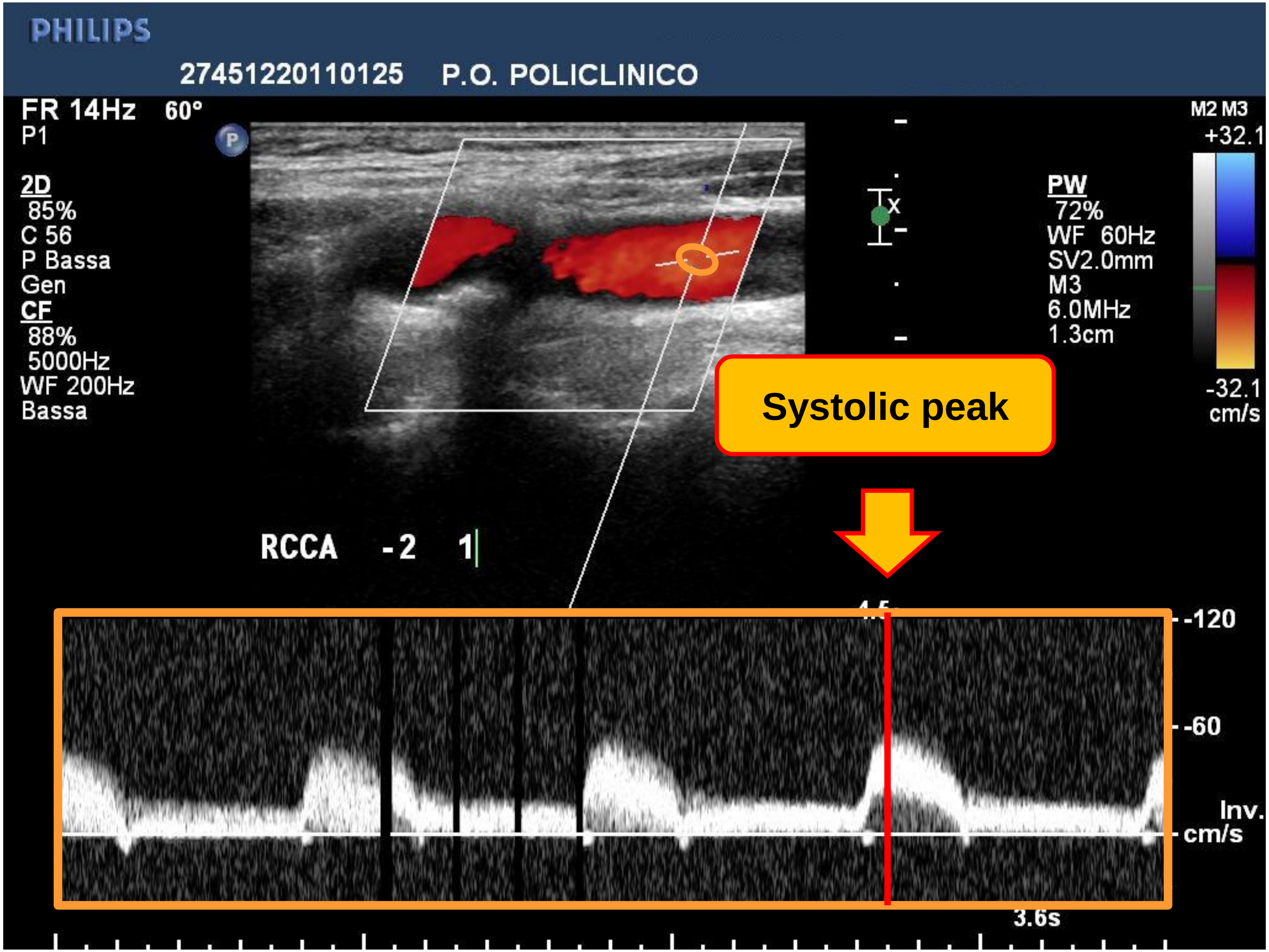
MOTIVATING PROBLEM: DATA



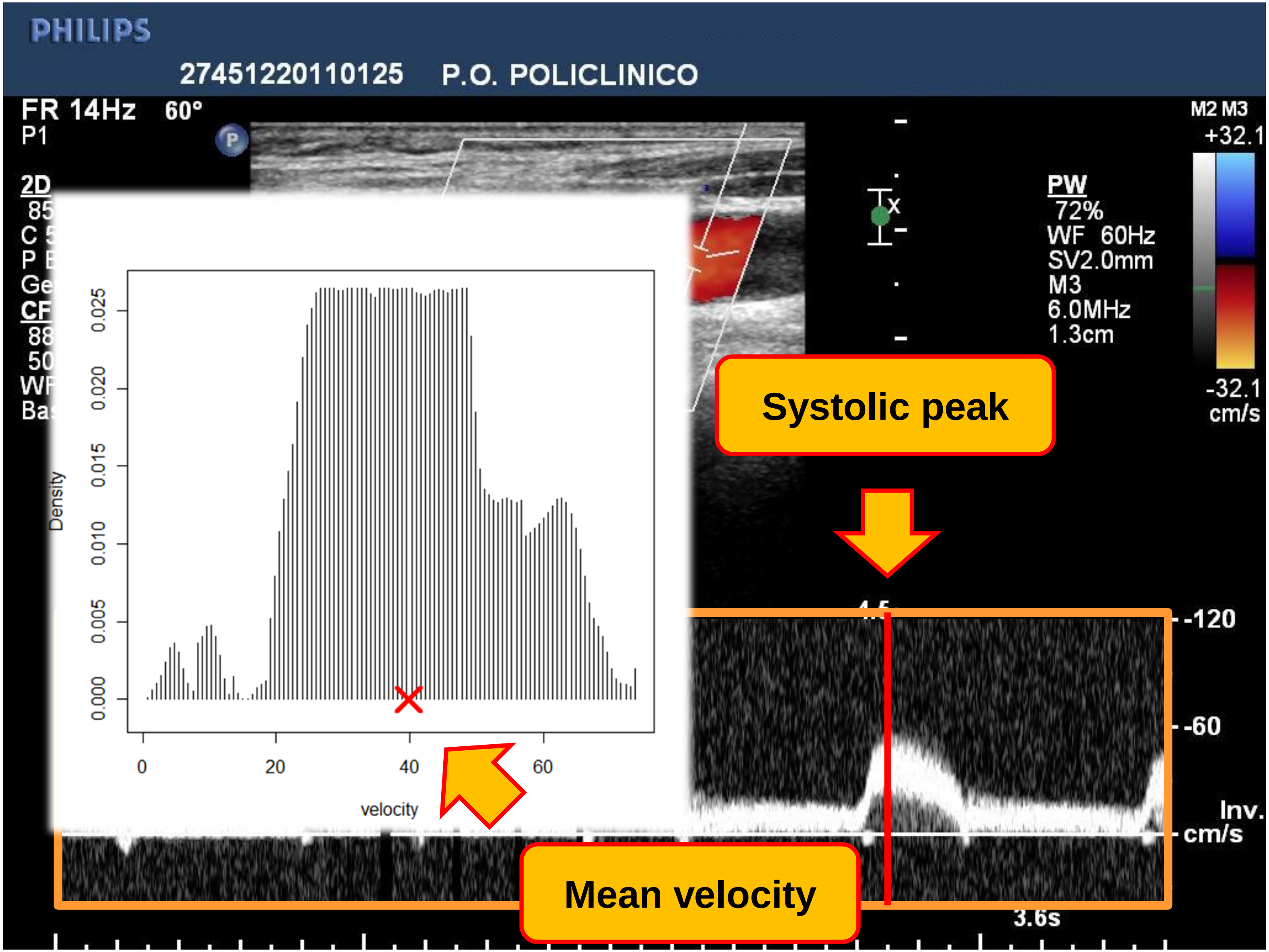
MOTIVATING PROBLEM: DATA



MOTIVATING PROBLEM: DATA



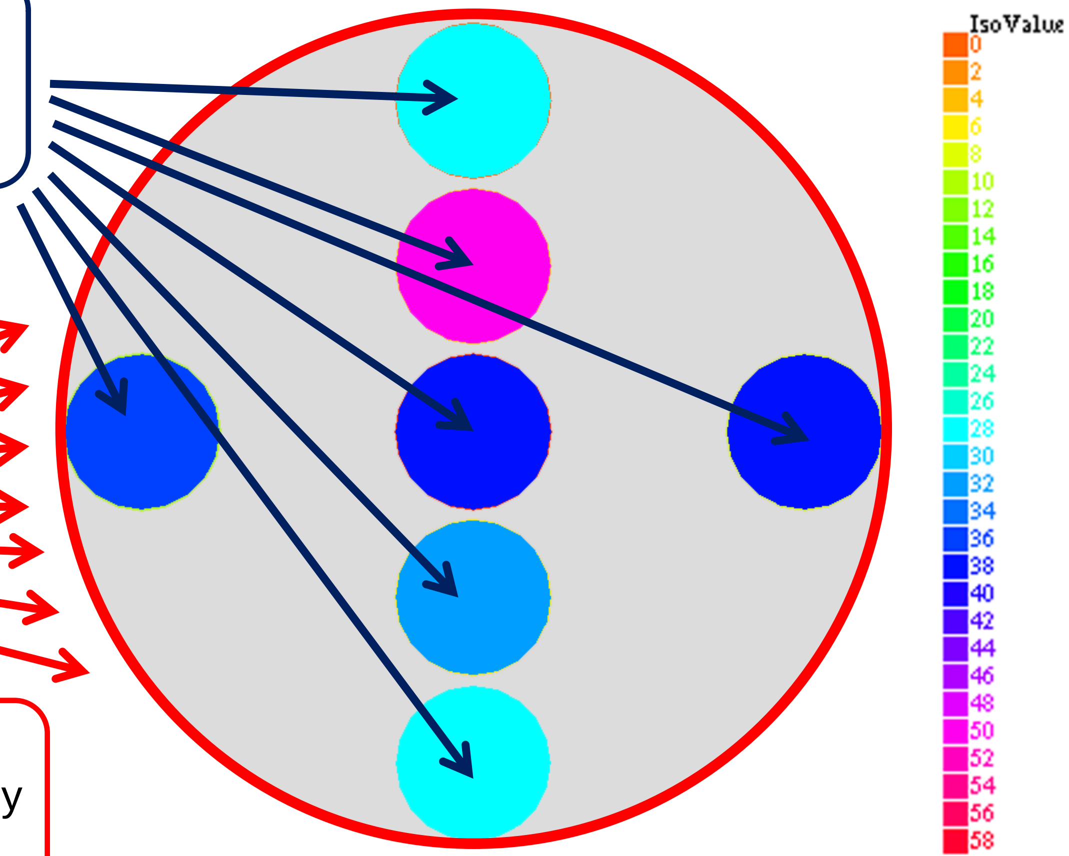
MOTIVATING PROBLEM: DATA



MOTIVATING PROBLEM: DATA

7 beams:
mean velocity
over the beam

Physiological boundary conditions: velocity=0 near the arterial wall



The theoretical solution for a stationary velocity field in a perfectly straight pipe is a parabolic profile

Smoothing over an irregular domain with boundary conditions ⇨ Classic smoothing techniques are not fit for the purpose

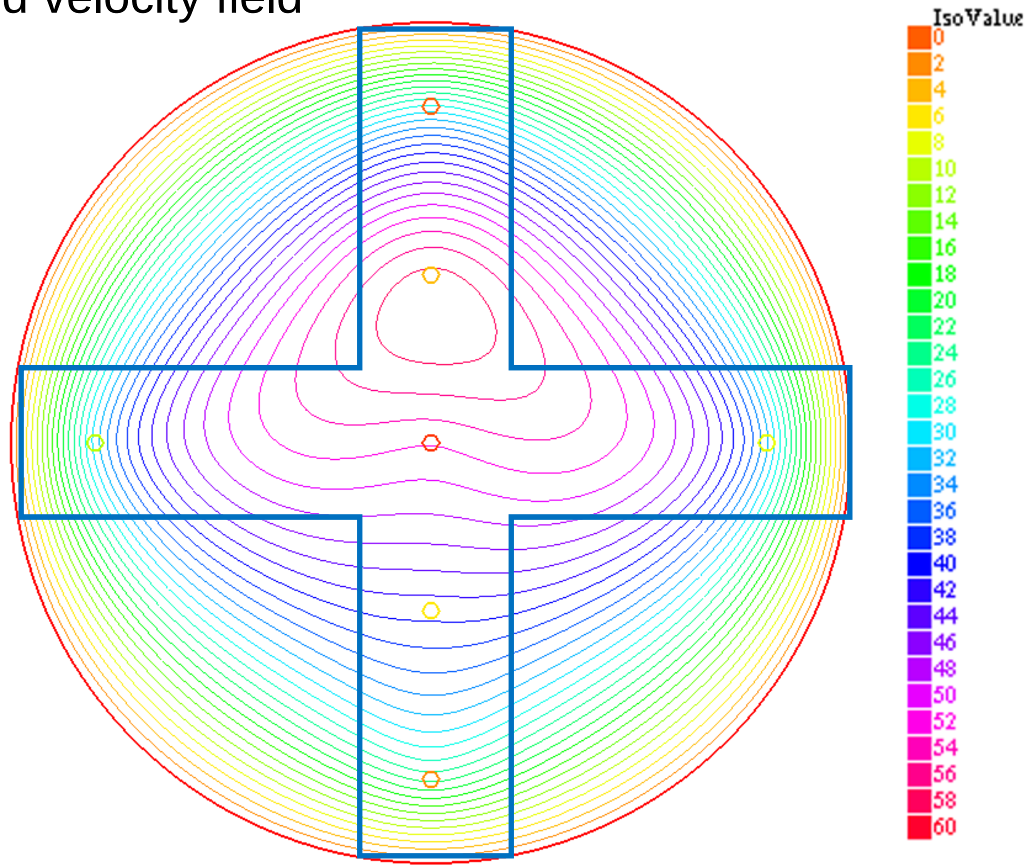
We use SSR to estimate the blood velocity field

minimize the functional

$$J(f) = \sum_{i=1}^{n} (f(\mathbf{p}_i) - z_i)^2 + \lambda \int_\Omega (\Delta f)^2$$

Non physiological velocity field

Squared isolines are caused by the cross-shaped pattern of the observations



EXTENTIONS of SSR models:

✓ inclusion of the physical or physiological knowledge of the phenomenon under study;

✓data distributed over subregions.

MODEL FOR POINTWISE OBSERVATIONS

Let $\Omega \subset R^2$ be a bounded and regular domain and z_i the n observations located at points $\mathbf{p}_i \in \Omega$.

We consider the model $$z_i = f(\mathbf{p}_i) + \epsilon_i$$

$\boldsymbol{\varepsilon}_i$: independent errors

$E[\underline{\boldsymbol{\varepsilon}}_i]=0$, $Var(\boldsymbol{\varepsilon}_i)=\boldsymbol{\sigma}^2$

The surface f: $\Omega \rightarrow R$ can be estimated minimizing the penalized least square functional:

$$J(f) = \sum_{i=1}^{n} (f(\mathbf{p}_i) - z_i)^2 + \lambda \int_\Omega (Lf - u)^2$$

misfit of a PDE that models the phenomenon under study

over the space of surfaces $f \in H^2(\Omega)$ that satisfy the required boudary conditions on $\partial\Omega$

L is a second order elliptic operator $Lf = -\mathrm{div}(K\nabla f) + \mathbf{b} \cdot \nabla f + cf$

$u \in L^2(\Omega)$ is the forcing term

The parameters can be spatially dependent

Problem (*)

Mininimize the functional:

$$J(f) = \sum_{i=1}^{n} (f(\mathbf{p}_i) - z_i)^2 + \lambda \int_{\Omega} (Lf - u)^2$$

over the space of surfaces f $\in$ $H^2(\Omega)$ with proper boundary conditions.

Proposition 1

The solution of the problem (*) exists and is unique. The solution is obtained solving in a weak sense the two problems:

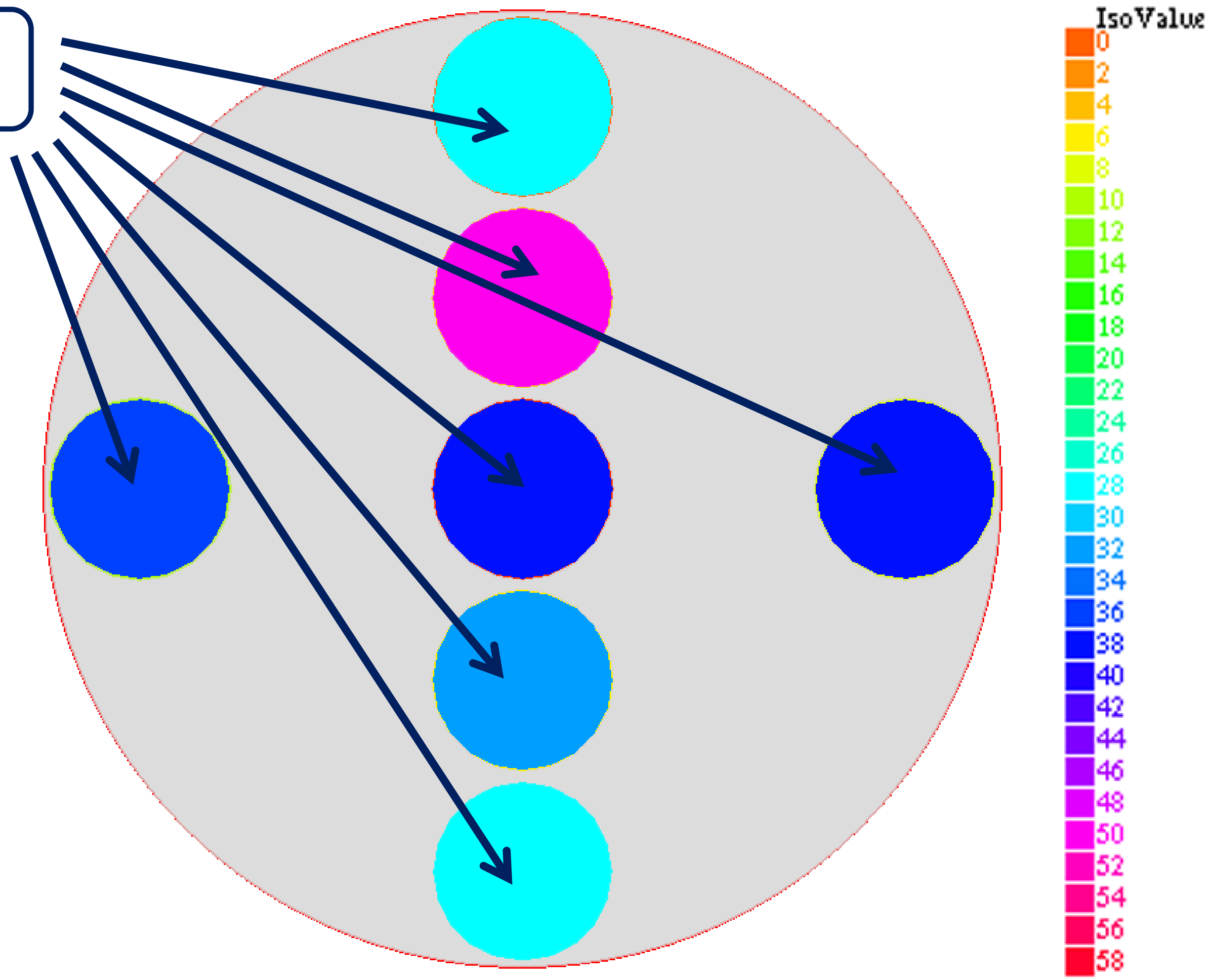
$$\begin{cases} L\hat{f} = u + \hat{g} & in\ \Omega \\ +\ b.c. & on\ \partial\Omega \end{cases} \qquad \begin{cases} L^*\hat{g} = -\frac{1}{\lambda}\sum_{i=1}^{n}(\hat{f}(\mathbf{p}) - z_i)\delta_{\mathbf{p}_i}(\mathbf{p}) & in\ \Omega \\ +\ b.c. & on\ \partial\Omega \end{cases}$$

where L* is the adjoint operator of L.

The proof is based on the convexity of the functional *J*, PDE optimal control theory and regularity of the solution of elliptic PDEs.

In the eco-doppler application we don't have pointwise data but only data over some subdomains $D_i \subset \Omega$ for i=1,…n.

7 subdomains



MODEL FOR AREAL DATA

In the eco-doppler application we don't have pointwise data but only data over some subdomains $D_i \subset \Omega$ for i=1,…n.

We consider the model $$z_{ij} = f(\mathbf{p}_{ij}) + \epsilon_{ij}$$

ε_i: independent errors

$E[\varepsilon_i]=0$, $Var(\varepsilon_i)=\sigma^2$

where z_{ij} for i=1,…,n and j=1,…,m are the observations located at points $\mathbf{p}_{ij} \in D_i \subset \Omega$.

The location points $\mathbf{p}_{ij}$ are unknown
we only know that $\mathbf{p}_{ij} \in D_i$

We consider only $\bar{z}_i$
the mean value over the subdomain D_i

The model that we consider for the mean value over the subdomain i $\bar{z}_i$ is:

$$\bar{z}_i = \frac{1}{|D_i|}\int_{D_i} f(\mathbf{p})d\mathbf{p} + \eta_i \qquad \eta_i = \frac{1}{m}\sum_{j=1}^{m}\epsilon_{ij}$$

$E[\eta_i]=0$, $Var(\eta_i)=\sigma^2/m=\sigma_\eta^2$

The model is obtained approximating $\frac{1}{m_i}\sum_{j=1}^{m_i} f(\mathbf{p}_{ij}) \approx \mathbb{E}[f(P)|\, P \in D_i]$

MODEL FOR AREAL DATA

The surface f: $\Omega \to R$ is estimated minimizing, over the space of surfaces $f \in H^2(\Omega)$ that satisfy the required boudary conditions on $\partial\Omega$, the penalized least square functional:

$$J(f) = \sum_{i=1}^{n} \left(\int_{D_i} (f(\mathbf{p}) - \bar{z}_i)\, d\mathbf{p} \right)^2 + \lambda \int_\Omega (Lf - u)^2 \qquad (**)$$

weighted least-square-error functional
for the areal mean over subdomains D_i

Proposition 2

The problem of minimizing (**) is well posed, the solution exists and is unique.
The solution is obtained solving in a weak sense the two problems:

$$\begin{cases} L\hat{f} = u + \hat{g} & in\ \Omega \\ +\ b.c. & on\ \partial\Omega \end{cases} \qquad \begin{cases} L^*\hat{g} = -\frac{1}{\lambda} \sum_{i=1}^{n} \mathbb{I}_{D_i}(\mathbf{p}) \int_{D_i} (\hat{f}(\mathbf{p}_i) - \bar{z}_i) d\mathbf{p} & in\ \Omega \\ +\ b.c. & on\ \partial\Omega \end{cases}$$

where L* is the adjoint operator of L.

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PROPERTIES OF THE ESTIMATOR

In spatial regression models with a partial differential regularization the surface estimators have good asymptotic properties:

• Consistency (Infill asymptotic)

Proposition 3

When $Lf=\Delta f$ the surface estimator $\hat{f}$ is consistent and converges almost surely to f in the L^2 norm for $n \to +\infty$ and $\lambda=\lambda(n) \sim n^{\beta}$ with $1/2<\beta<1$.

We can write the estimator $\hat{f} = f^* + w$ such that $\mathbb{E}\left[\hat{f}\right] = f^*$

deterministic part (f^*), random part (w)

The consistency is proved thanks to the error decomposition

$$||\hat{f} - f||_{L^2} \leq ||f^* - f||_{L^2} + ||\hat{f} - f^*||_{L^2}$$

if $\lambda/n \to 0$: $||f^* - f||_{L^2} \to 0$

$||\hat{f} - f^*||_{L^2} \to 0$ thanks to the Ascoli-Arzela theorem in the weak topology and the law of large numbers

PROPERTIES OF THE ESTIMATOR

In spatial regression models with a partial differential regularization the surface estimators have good asymptotic properties:

• Consistency (Infill asymptotic)

Proposition 3

When $Lf=\Delta f$ the surface estimator $\hat{f}$ is consistent and converges almost surely to f in the L^2 norm for $n \to +\infty$ and $\lambda=\lambda(n) \sim n^{\beta}$ with $1/2<\beta<1$.

WORK IN PROGRESS: consistency of the estimator $\hat{f}$ when L is an elliptic operator

• Asymptotic normality (work in progress)

The estimator is linear in the observations but it is defined in an implicit form

$$\sum_{i=1}^{n} \hat{f}(\mathbf{p}_i)v(\mathbf{p}_i) + \lambda \int L\hat{f}Lv = \sum_{i=1}^{n} z_i v(\mathbf{p}_i) \quad \forall v \in H^2$$

The solution $\hat{f}$ is not analytically computable and in practice it is approximated by means of the Mixed Finite Elements method.

Proposition 4

When $Lf=\Delta f$, the discrete surface estimator $\hat{f}_h$ is consistent and converges almost surely to f in the L^2 norm for $n \to +\infty$, the dimension of the mesh $h \to 0$ and under the same conditions of the continuos estimator $\hat{f}$.

The convergence is proved for every polynomial order $p \geq 1$ of the Finite Elements approximation.

The consistency is proved thanks to the error decomposition

$$\left\| \hat{f}_h - f \right\|_{L^2} \leq \left\| \hat{f}_h - \hat{f} \right\|_{L^2} + \left\| \hat{f} - f \right\|_{L^2}$$

discretization error — intrinsic error

The evaluation of the discrete surface estimator $\hat{f}_h$ on a set of points $\xi_1,\ldots,\xi_m$

$\hat{\mathbf{f}}_h=(\hat{f}_h(\xi_1),\ldots,\hat{f}_h(\xi_m))$ is a linear function of the vector of observations $\mathbf{z}=(z_1,\ldots,z_n)$

$$\hat{\mathbf{f}}_h = C\mathbf{z} + \mathbf{d}$$

C is the matrix obtained by the EF discretization of the problem.

$\mathbf{d}$ is a constant vector.

We can compute classic inferential tool (pointwise confidence bands or prediction intervals) based on the computation of the covariance matrix:

$$\mathbb{C}\text{ov}(\hat{\mathbf{f}}_h) = \sigma^2 C C^T$$

where σ^2 is the error variance.

Bias and variance of the surface estimator are strongly influenced by the penalized differential operator.

APPLICATION TO THE BLOOD VELOCITY PROFILE ESTIMATION:

Thanks to the physical and physiological knowledge of the problem we can choose the parameters of the elliptic operator L:

$$Lf = -\mathrm{div}(K\nabla f) + \mathbf{b}\cdot\nabla f + cf$$
$$f|_{\Omega} = 0$$

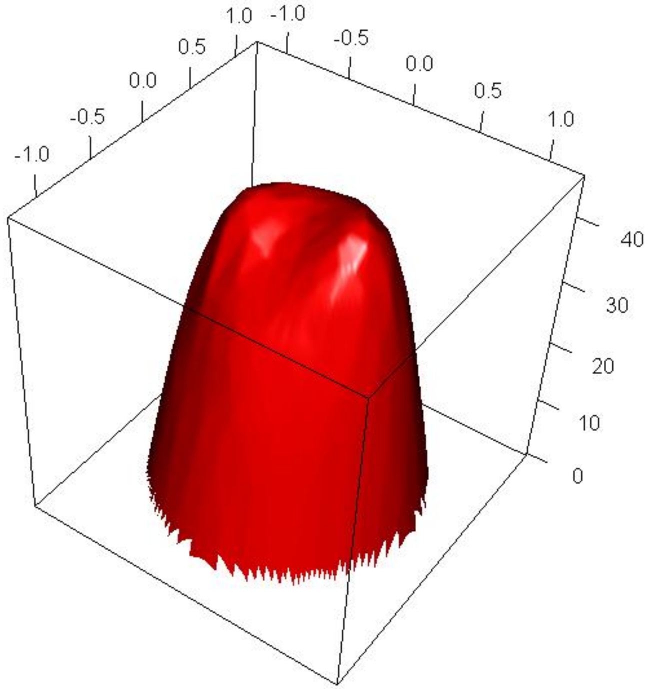
Anisotropic diffusion that smooths the observations along circles

$$K(\mathbf{x}) = \begin{bmatrix} \nu_1 x_2^2 + \nu_2 x_1^2 & (\nu_2-\nu_1)x_1x_2 \\ (\nu_2-\nu_1)x_1x_2 & \nu_1 x_1^2 + \nu_2 x_2^2 \end{bmatrix} + \nu_3\left(R^2 - x_1^2 - x_2^2\right) I$$

Radial transport field that smooths the observations along the radial direction

$$\mathbf{b}(\mathbf{x}) = (x_1, x_2)^T$$

$$c = 0$$



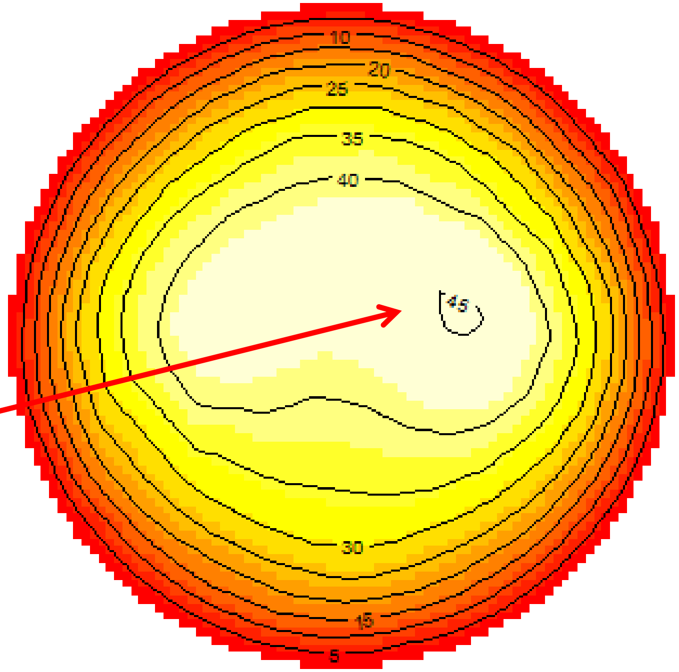
Blood velocity profile estimation

Estimated velocity field

Velocity profile similar to a parabolic profile

theoretical solution for a stationary velocity field in a perfectly straight pipe

Asymmetry due to the geometry of the common carotid artery



Richer information than the original observations
(shape of the velocity profile and some features i.e. eccentricity)

Study of the interactions between blood fluid-dynamics and presence and properties of atherosclerotic plaques

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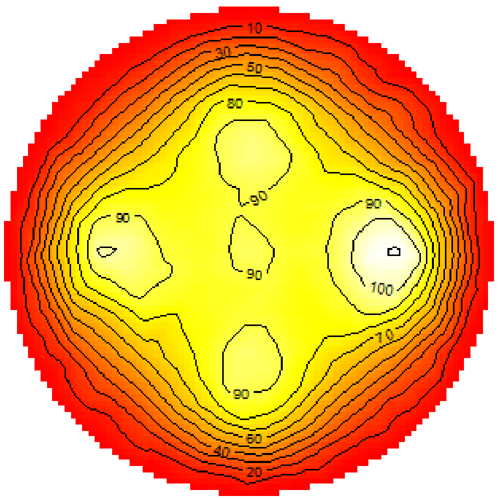
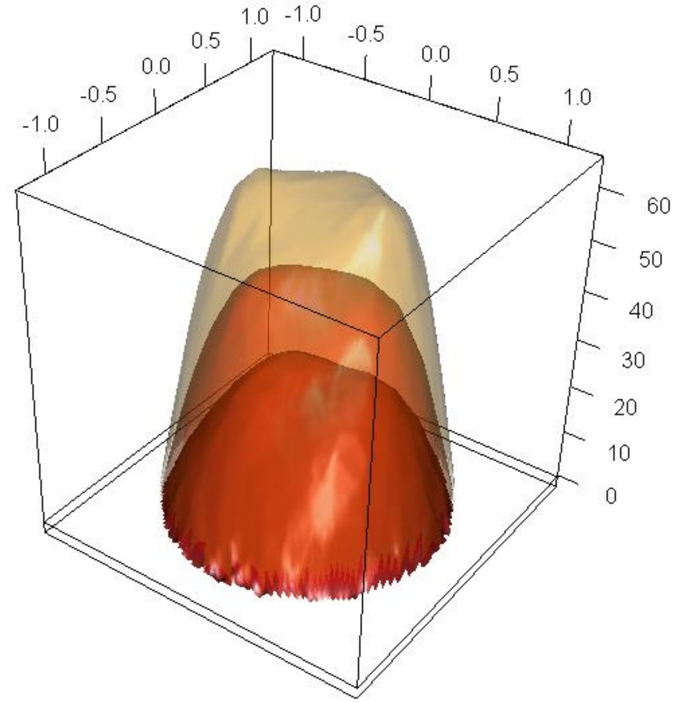
BLOOD VELOCITY PROFILE ESTIMATION

Computing the variance of the surface estimator we can obtain pointwise confidence bands for the velocity profile.



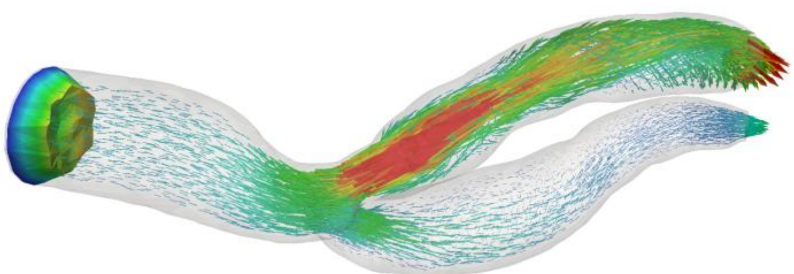
variance



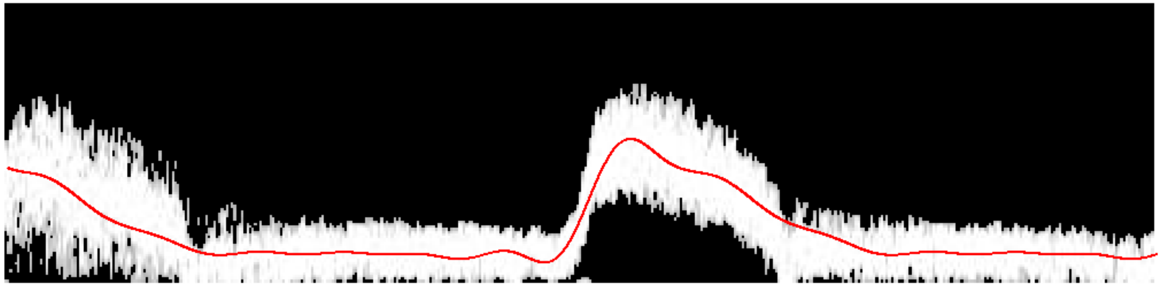
pointwise 0.95 confidence bands

Patient-specific inflow conditions for CFD

Effect of the misspecification of the inflow conditions on the results of the blood-flow numerical simulations



For each subdomain we consider the mean velocity varying in time

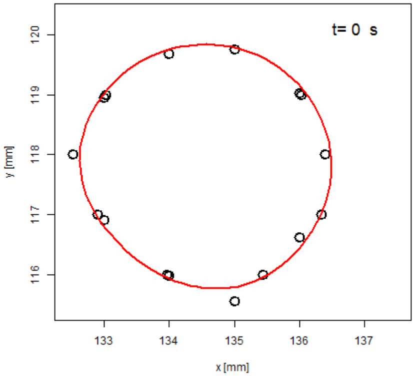


Study of the variations of the blood-flow velocity over the time of the heart beat

We can estimate the surface f: $(\Omega, [0,T]) \rightarrow$ R minimizing the penalized functional:

$$J(f) = \sum_{i=1}^{n} \sum_{j=1}^{n_t} (f(\mathbf{p}_i, t_j) - z_{ij})^2 + \lambda \int_0^T \int_\Omega \left(\frac{\partial f}{\partial t} + Lf - u \right)^2$$

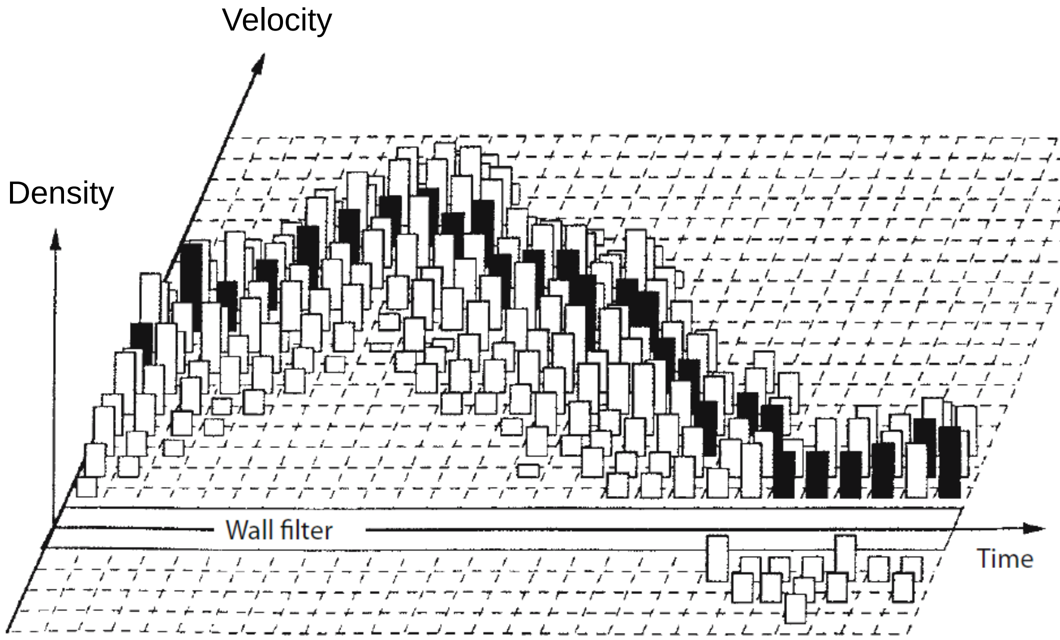
Penalization of a parabolic PDE



Geometry of the domain that changes in time

recostruction from MRI data by Elena Faggiano

The eco-doppler signal represents the histogram of the velocities of the blood particles in the beam varying in time



For each subdomain we have the histogram of the observations of the response variable but not the location points

Estimate a surface with spatially distributed histograms

MODEL EXTENTIONS: ASYMPTOTIC PROPERTIES

1. Study of the asymptotic properties of the surface estimator

   - Consistency (Infill asymptotic):

     when $Lf=\Delta f$ the surface estimator $\hat{f}$ is consistent for $n \to +\infty$ and $\lambda=\lambda(n)$.

     WORK IN PROGRESS: consistency when the penalty is an elliptic PDE
     consistency of the Finite Element estimator

   - Asymptotic normality

2. Interpretation of the roughness penality term as a covariance structure for the spatial dependence

3. Estimation of the parameters K, b, c of the penalized PDE from the data (extension of Ramsay et al, 2007, JRSSB).

SOME REFERENCES

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Simulation performed in FreeFem++ and R.
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